System of equations involving determinant.

https://www.linkedin.com/feed/update/urn:li:activity:6532446281925832704

Solve for real numbers:

$$\begin{cases} \det\begin{pmatrix} x & y & 2 & 3 \\ y & x & 3 & 2 \\ 2 & 3 & x & y \\ 3 & 2 & y & x \end{pmatrix} = 0 \\ x + y - \sqrt{xy} = \sqrt{\dfrac{x^2+y^2}{2}} \end{cases}.$$

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First note that x, y must be nonnegative, because $xy \geq 0$ and

$$x + y = \sqrt{xy} + \sqrt{\frac{x^2+y^2}{2}} \geq 0.$$

Using transformation $R_1 := R_1 + R_2 + R_3 + R_4$ we obtain

$$D(x,y) := \det\begin{pmatrix} x & y & 2 & 3 \\ y & x & 3 & 2 \\ 2 & 3 & x & y \\ 3 & 2 & y & x \end{pmatrix} = \det\begin{pmatrix} x+y+5 & x+y+5 & x+y+5 & x+y+5 \\ y & x & 3 & 2 \\ 2 & 3 & x & y \\ 3 & 2 & y & x \end{pmatrix} =$$

$$(x+y+5)\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ y & x & 3 & 2 \\ 2 & 3 & x & y \\ 3 & 2 & y & x \end{pmatrix}.$$

Using transformations $C_i := C_i - C_1, i = 2,3,4$ we obtain

$$\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ y & x & 3 & 2 \\ 2 & 3 & x & y \\ 3 & 2 & y & x \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 0 & 0 \\ y & x-y & 3-y & 2-y \\ 2 & 1 & x-2 & y-2 \\ 3 & -1 & y-3 & x-3 \end{pmatrix} =$$

$$\det\begin{pmatrix} x-y & 3-y & 2-y \\ 1 & x-2 & y-2 \\ -1 & y-3 & x-3 \end{pmatrix} \overset{R_1:=R_1+R_2}{=} \det\begin{pmatrix} x-y+1 & x-y+1 & 0 \\ 1 & x-2 & y-2 \\ -1 & y-3 & x-3 \end{pmatrix} =$$

$$\overset{C_2:=C_2-C_1}{=} \det\begin{pmatrix} x-y+1 & 0 & 0 \\ 1 & x-2 & y-2 \\ -1 & y-3 & x-3 \end{pmatrix} = (x-y+1)\det\begin{pmatrix} 1 & 0 & 0 \\ 1 & x-3 & y-2 \\ -1 & y-2 & x-3 \end{pmatrix} =$$

$$(x-y+1)\det\begin{pmatrix} x-3 & y-2 \\ y-2 & x-3 \end{pmatrix} = (x-y+1)(x^2-6x-y^2+4y+5) = \left((x-y)^2-1\right).$$

Let $u := x+y, v := x-y$. Then $D(x,y) = \left((x+y)^2-25\right)\left((x-y)^2-1\right) = (u^2-25)(v^2-1)$

and the system $\begin{cases} D(x,y) = 0 \\ x+y-\sqrt{xy} = \sqrt{\dfrac{x^2+y^2}{2}} \end{cases}$ becomes

(**1**) $\begin{cases} (u^2-25)(v^2-1) = 0 \\ u-\sqrt{\dfrac{u^2-v^2}{4}} = \sqrt{\dfrac{u^2+v^2}{4}} \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} u = 5 \\ |v| = 1 \end{array}\right. \\ u-\sqrt{\dfrac{u^2-v^2}{4}} = \sqrt{\dfrac{u^2+v^2}{4}} \end{cases}.$

In the case $u = 5$ we obtain

$5-\sqrt{\dfrac{25-v^2}{4}} = \sqrt{\dfrac{25+v^2}{4}} \Leftrightarrow \sqrt{25+v^2}+\sqrt{25-v^2} = 10 \overset{v\geq 0}{\Leftrightarrow} v = 0$ and, therefore,

$\begin{cases} x+y = 5 \\ x = y \end{cases} \Leftrightarrow x = y = 2.5;$

In the case $|v| = 1$ we obtain equation

$u-\sqrt{\dfrac{u^2-1}{4}} = \sqrt{\dfrac{u^2+1}{4}} \Leftrightarrow 2u = \sqrt{u^2+1}+\sqrt{u^2-1}$

which have no solutions in $\mathbb{R}$ because $2u = \sqrt{u^2+1}+\sqrt{u^2-1} \Leftrightarrow u^2 = \sqrt{u^4-1}$.

Thus, final answer is $x = y = \dfrac{5}{2}$.